

RELATIVE DIFFERENTIAL COHOMOLOGY

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ABSTRACT. Let h^\bullet be a rationally even cohomology theory and \hat{h}^\bullet the natural differential refinement, as defined by Hopkins and Singer. We consider the possible definitions of the relative differential cohomology groups, generalizing the analogous picture for the Deligne cohomology, and we show the corresponding long exact sequence in each case.

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1. INTRODUCTION

Let h^\bullet be a rationally even cohomology theory and \hat{h}^\bullet the natural differential refinement, as defined in [12, 14]. In [14] a relative version of \hat{h}^\bullet has been defined, considering classes on a pair of manifolds (X, A) whose restriction to A is trivial: nevertheless, when considering the differential refinement of ordinary cohomology, this version is not the only interesting one. It is more natural to consider the case in which the restriction to A is only topologically trivial, so that the curvature is exact but not necessarily vanishing, as in the first definition of relative Cheeger-Simons character given in [3]. For example, the B -field in string theory can be described as a Deligne cohomology class of degree 2 in the space-time, which is topologically trivial on a spin^c D-brane world-volume, because of the Freed-Witten anomaly [10, 2, 7]. In this case, imposing that the B -field is vanishing on the world-volume is an unnecessary restriction. Moreover, in [3] the authors showed that it is possible to define the relative Cheeger-Simons characters in such a way that they fit into a long exact sequence completely made by differential cohomology groups, and it is natural to inquire if such a definition can be generalized.

We have shown in [9] that there are actually four possible inequivalent ways to define the relative Deligne cohomology groups, two of which being the most meaningful and corresponding to the two kinds of differential characters defined in [3]. We generalize this picture to any cohomology theory and we show in each case the corresponding long exact sequence. Such a generalization can be applied for example to describe the B -field when it is thought of as an element of a generalized cohomology theory different from the ordinary one [6], or any background field that must be topologically trivial on a subspace in order to delete an anomaly.

The paper is organized as follows. In section 2 we recall the possible definitions of the relative Deligne cohomology groups. In section 3 we generalize such definitions to any cohomology theory for a pair of manifolds and we show the corresponding long exact sequences. In section 4 we construct the Bockstein map of each exact sequence and we prove the exactness. In section 5 we generalize the previous constructions to any map of manifolds, not necessarily an embedding.

2. RELATIVE DELIGNE COHOMOLOGY

We briefly recall the four possible versions of relative Deligne cohomology, as shown in detail in [9]. A Deligne cohomology class [4] of degree p on a smooth manifold X is a hypercohomology class of degree p of the complex of sheaves:

$$(1) \quad \underline{U}(1) \xrightarrow{\tilde{d}} \Omega_{\mathbb{R}}^1 \xrightarrow{d} \cdots \xrightarrow{d} \Omega_{\mathbb{R}}^p,$$

where $\underline{U}(1)$ is the sheaf of smooth $U(1)$ -valued functions, $\Omega_{\mathbb{R}}^k$ is the sheaf of smooth real differential forms of degree k and $\tilde{d} = \frac{1}{2\pi i} d \circ \log$. The curvature is a globally defined closed differential form of degree $p+1$. Given a smooth map $f : A \rightarrow X$ of manifolds without boundary, we consider the complex (1) in degree p on X and q on A , which we call:

$$(2) \quad S_{X,p}^{\bullet} := \underline{U}(1)_X \xrightarrow{\tilde{d}} \cdots \xrightarrow{d} \Omega_{X,\mathbb{R}}^p \quad S_{A,q}^{\bullet} := \underline{U}(1)_A \xrightarrow{\tilde{d}} \cdots \xrightarrow{d} \Omega_{A,\mathbb{R}}^q.$$

We consider the push-forward $f_* S_{A,q}^{\bullet}$, recalling that, for \mathcal{F} a sheaf on A , the sheaf $f_* \mathcal{F}$ on X is defined as $(f_* \mathcal{F})(U) := \mathcal{F}(f^{-1}U)$ for any $U \subset X$ open. For $q \leq p$ there is a natural map of complexes of sheaves on X :

$$(3) \quad \varphi_{f,p,q} : S_{X,p}^{\bullet} \rightarrow f_* S_{A,q}^{\bullet},$$

defined in the following way: in degree $i \leq q$ it pulls back via f from U to $f^{-1}U$ the function or differential form in the domain, in degree $i > q$ it is the zero-map.¹ One can construct the mapping cone of the morphism (3), i.e. the complex of sheaves on X [11]:²

$$(4) \quad C(\varphi_{f,p,q})^{\bullet} := S_{X,p}^{\bullet} \oplus f_* S_{A,q}^{\bullet-1} \quad d_{C(\varphi_{f,p,q})}^{\bullet} := \begin{bmatrix} d_{S_{X,p}}^{\bullet} & 0 \\ \varphi_{f,p,q}^{\bullet} & -f_* d_{S_{A,q}}^{\bullet-1} \end{bmatrix}.$$

Definition 2.1. *The relative Deligne cohomology groups of type (p, q) of the map $f : A \rightarrow X$ are the hypercohomology groups of (4). We denote them by $H^n(S_{X,p}, S_{A,q}, f)$.*

2.1. Groups of type I. For $p = q$ the long exact sequence in cohomology corresponds, up to canonical isomorphism, to the following sequence (we denote by $H^{\bullet}(X, A, f; G)$ the relative cohomology groups with coefficients in G associated to the map f):

$$(5) \quad \begin{aligned} \cdots &\longrightarrow H^{p-1}(X, A, f; \mathbb{R}/\mathbb{Z}) \longrightarrow H^{p-1}(X; \mathbb{R}/\mathbb{Z}) \longrightarrow H^{p-1}(A; \mathbb{R}/\mathbb{Z}) \\ &\longrightarrow H^p(S_{X,p}, S_{A,p}, f) \longrightarrow H^p(S_{X,p}) \longrightarrow H^p(S_{A,p}) \\ &\longrightarrow H^{p+2}(X, A, f; \mathbb{Z}) \longrightarrow H^{p+2}(X; \mathbb{Z}) \longrightarrow H^{p+2}(A; \mathbb{Z}) \longrightarrow \cdots \end{aligned}$$

A class in $H^p(S_{X,p}, S_{A,p}, f)$ is trivial as a differential class when pulled back on A , therefore this group corresponds, with respect to ordinary cohomology, to the one considered in [14].

2.2. Groups of type II. For $q = p-1$ the long exact sequence in cohomology corresponds, up to canonical isomorphism, to the following sequence:

$$(6) \quad \begin{aligned} \cdots &\longrightarrow H^{p-1}(X, A, f; \mathbb{R}/\mathbb{Z}) \longrightarrow H^{p-1}(X; \mathbb{R}/\mathbb{Z}) \longrightarrow H^{p-1}(S_{A,p-1}) \\ &\longrightarrow H^p(S_{X,p}, S_{A,p-1}, f) \longrightarrow H^p(S_{X,p}) \longrightarrow H^{p+1}(A; \mathbb{Z}) \\ &\longrightarrow H^{p+2}(X, A, f; \mathbb{Z}) \longrightarrow H^{p+2}(X; \mathbb{Z}) \longrightarrow H^{p+2}(A; \mathbb{Z}) \longrightarrow \cdots \end{aligned}$$

¹It is necessary that $q \leq p$, otherwise (3) would not be a map of complexes, since it would not commute with the coboundary of degree p .

²The matrix defining the coboundary is supposed to multiply from the left a column vector.

A class in $H^p(S_{X,p}, S_{A,p}, f)$ is only *topologically* trivial when pulled back on A , therefore the curvature is exact, but not necessarily vanishing on A (it must vanish as a de-Rham cohomology class, not as a single form, but the curvature is meaningful as a representative). Such a class is represented on A by a cocycle of the form $(1, 0, \dots, 0, \rho)$, where $d\rho$ is the curvature on A ; in other words, a specific trivialization is chosen on A . This group corresponds to the one used in [1, 13, 7, 2] in order to describe the B -field in string theory. There is a natural embedding:

$$(7) \quad \varphi_{f,p,p-1} : H^p(S_{X,p}, S_{A,p}, f) \hookrightarrow H^p(S_{X,p}, S_{A,p-1}, f).$$

2.3. Groups of type III. For $q \leq p-2$ the long exact sequence in cohomology, starting from two positions before $H^p(S_{X,p}, S_{A,q}, f)$, always corresponds, up to canonical isomorphism, to the following sequence:

$$(8) \quad \begin{aligned} & \dots \longrightarrow H^{p-1}(X; \mathbb{R}/\mathbb{Z}) \longrightarrow H^p(A; \mathbb{Z}) \\ & \longrightarrow H^p(S_{X,p}, S_{A,q}, f) \longrightarrow H^p(S_{X,p}) \longrightarrow H^{p+1}(A; \mathbb{Z}) \\ & \longrightarrow H^{p+2}(X, A, f; \mathbb{Z}) \longrightarrow H^{p+2}(X; \mathbb{Z}) \longrightarrow H^{p+2}(A; \mathbb{Z}) \longrightarrow \dots \end{aligned}$$

This is the less interesting case. We consider for simplicity $q = 0$. A class in $H^p(S_{X,p}, S_{A,0}, f)$ is again only *topologically* trivial when pulled back on A , therefore the curvature is exact, but not necessarily vanishing, on A . Nevertheless, there is not a preferred trivialization. There is a natural surjective map:

$$(9) \quad \varphi_{f,p-1,0} : H^p(S_{X,p}, S_{A,p-1}, f) \rightarrow H^p(S_{X,p}, S_{A,0}, f).$$

The first part of the long exact sequence actually depends on q .

2.4. Groups of type IV. In [3] the authors consider the relative version of the Hopkins-Singer groups [12]. In particular, they start from the complex:

$$\check{C}^\bullet(X) := C^\bullet(X; \mathbb{Z}) \oplus C^{\bullet-1}(X; \mathbb{R}) \oplus \Omega_{int}^\bullet(X),$$

where $\Omega_{int}^\bullet(X)$ is the group of closed forms representing an integral cohomology class, with coboundary $\check{\delta}(c, h, \omega) := (\delta c, \omega - c - \delta h, 0)$. Then they construct the mapping cone of this complex with respect to a smooth map $f : A \rightarrow X$, i.e. they consider:

$$\check{C}^\bullet(f) := \check{C}^\bullet(X) \oplus \check{C}^{\bullet-1}(A)$$

with differential $\delta(S, T) = (\delta S, f^*S - \delta T)$. The relative Hopkins-Singer groups are the cohomology groups of such a complex. By construction they fit into a long exact sequence completely made by differential cohomology groups, while the previous ones contain some differential and some topological cohomology groups. We can recover such a definition, in particular such a long exact sequence, within the language of Deligne cohomology, even if the definition is more artificial. Actually, we will show that the generalization to any cohomology theory can be defined in a quite natural way, therefore the fact of being artificial only concerns Deligne cohomology. We consider the following maps:

- $H^p(S_{X,p}, S_{A,p-1}, f) \rightarrow H^p(S_{A,p})$, defined composing the map $H^p(S_{X,p}, S_{A,p-1}, f) \rightarrow H^p(S_{X,p})$ appearing in the sequence (6) with the restriction map $H^p(S_{X,p}) \rightarrow H^p(S_{A,p})$ appearing in the sequence (5);
- $H^{p-1}(S_{X,p-1}) \rightarrow H^p(S_{X,p}, S_{A,p-1}, f)$, defined composing the restriction map $H^{p-1}(S_{X,p-1}) \rightarrow H^{p-1}(S_{A,p-1})$ appearing in the sequence (5) with the Bockstein map $H^{p-1}(S_{A,p-1}) \rightarrow H^p(S_{X,p}, S_{A,p-1}, f)$ appearing in the sequence (6).

We define:

$$(10) \quad \overline{H}^p(S_{X,p}, S_{A,p}, f) := \frac{\text{Ker}(H^p(S_{X,p}, S_{A,p-1}, f) \rightarrow H^p(S_{A,p}))}{\text{Im}(H^{p-1}(S_{X,p-1}) \rightarrow H^p(S_{X,p}, S_{A,p-1}, f))}.$$

Clearly for $A = \emptyset$ we get $\overline{H}^p(S_{X,p}) \simeq H^p(S_{X,p})$. The following sequence is exact:

$$(11) \quad \cdots \longrightarrow H^{p-1}(S_{A,p-1}) \longrightarrow \overline{H}^p(S_{X,p}, S_{A,p}, f) \longrightarrow H^p(S_{X,p}) \longrightarrow H^p(S_{A,p}) \longrightarrow \cdots$$

as proven in [9, Theorem 2.1].

3. RELATIVE DIFFERENTIAL COHOMOLOGY

We fix a multiplicative cohomology theory h^\bullet which is rationally even, i.e. such that the odd-dimensional cohomology groups of the point are torsion or vanishing. We suppose that h^\bullet is represented by an Ω -spectrum $(E_n, e_n, \varepsilon_n)$, where e_n is the marked point of E_n and $\varepsilon_n : (\Sigma E_n, \Sigma e_n) \rightarrow (E_{n+1}, e_{n+1})$ is the structure map, whose adjoint $\tilde{\varepsilon}_n : E_n \rightarrow \Omega_{e_{n+1}} E_{n+1}$ is a homeomorphism (not only a homotopy equivalence). We also call $\mu_{n,m} : E_n \wedge E_m \rightarrow E_{n+m}$ the maps making E a ring spectrum (hence the theory h^\bullet multiplicative). Moreover, following [14], we fix the same data of [8, Section 2.3]. In particular, for $\{*\}$ a space with one point, we call:

$$\mathfrak{h}^\bullet := h^\bullet(\{*\}) \quad \mathfrak{h}_{\mathbb{R}}^\bullet := \mathfrak{h}^\bullet \otimes_{\mathbb{Z}} \mathbb{R} \quad \mathfrak{h}_{\mathbb{R}/\mathbb{Z}}^\bullet := h^\bullet(\{*\}; \mathbb{R}/\mathbb{Z}).$$

We now state the possible definitions of the relative differential cohomology groups and we show the corresponding long exact sequences, postponing the construction of the Bockstein maps and the proofs to the next section.

3.1. Relative differential cohomology of type I. This definition is the one considered in [14], and generalizes to any cohomology theory the first one of the previous section. We first recall the definition of differential function in the relative case [14].

Definition 3.1. *If (X, A) is a pair of smooth manifolds (even with boundary, in which case A is a neat submanifold), (Y, y_0) a topological space with marked point, V^\bullet a graded real vector space and $\kappa_n \in C^n(Y, y_0, V^\bullet)$ a real singular cocycle, a relative differential function from (X, A) to (Y, y_0, κ_n) is a triple (f, h, ω) such that:*

- $f : (X, A) \rightarrow (Y, y_0)$ is a continuous function;
- $h \in C^{n-1}(X, A; V^\bullet)$;
- $\omega \in \Omega_{cl}^n(X, A; V^\bullet)$ (i.e. $\omega|_A = 0$)

satisfying, for $\chi : \Omega^\bullet(X, A; V^\bullet) \rightarrow C^\bullet(X, A; V^\bullet)$ the natural homomorphism:

$$(12) \quad \delta^{n-1}h = \chi^n(\omega) - f^*\kappa_n.$$

Moreover, a homotopy between two relative differential functions (f_0, h_0, ω) and (f_1, h_1, ω) is a relative differential function $(F, H, \pi^*\omega) : (X \times I, A \times I) \rightarrow (Y, y_0, \kappa_n)$, such that F is a homotopy between f_0 and f_1 , $H|_{(X \times \{i\}, A \times \{i\})} = h_i$ for $i = 0, 1$, and $\pi : X \times I \rightarrow X$ is the natural projection.

Since a cochain $h \in C^{n-1}(X, A; V^\bullet)$ is uniquely represented by a cochain $h \in C^{n-1}(X; V^\bullet)$ such that $h|_A = 0$, the previous definition is equivalent to the following one:

Definition 3.2. *With the same data of definition 3.1, a relative differential function of type I from (X, A) to (Y, y_0, κ_n) is a triple (f, h, ω) such that:*

- (f, h, ω) is a differential function $(f, h, \omega) : X \rightarrow (Y, \kappa_n)$;

- $(f, h, \omega)|_A = (c_{y_0}, 0, 0)$,

where c_{y_0} is the constant map with value y_0 . Homotopies are defined as in definition 3.1.

It follows that a homotopy is constant on $A \times I$, i.e. $(F, H, \pi^*\omega)|_{A \times \{t\}} = (c_{e_n}, 0, 0)$ for any $t \in I$.

The group $\hat{h}_I^n(X, A)$ is defined in the following way:

- as a set, $\hat{h}_I^n(X, A)$ contains the homotopy classes of relative differential functions $(f, h, \omega) : (X, A) \rightarrow (E_n, e_n, \iota_n)$.
- The sum is defined by:

$$(13) \quad [(f, h, \omega)] + [(g, k, \rho)] := [(\alpha_n \circ (f, g), h + k + (f, g)^* A_{n-1}, \omega + \rho)].$$

- The first Chern class and the curvature are defined as $I[(f, h, \omega)] := [f]$ and $R[(f, h, \omega)] := \omega$, and the map $a : \Omega^{\bullet-1}(X, A; \mathfrak{h}_{\mathbb{R}}^{\bullet})/\text{Im}(d) \rightarrow \hat{h}_I^{\bullet}(X, A)$ (v. [5]) is defined as $a[\rho] := [c_{e_n}, \chi(\rho), d\rho]$, for c_{e_n} the constant map whose value is the marked point e_n .
- The S^1 -integration map and the product are defined similarly to the absolute case.

The corresponding long exact sequence is the following:

$$(14) \quad \begin{aligned} \cdots &\longrightarrow \hat{h}_{\mathfrak{h}}^{p-1}(X, A) \longrightarrow \hat{h}_{\mathfrak{h}}^{p-1}(X) \longrightarrow \hat{h}_{\mathfrak{h}}^{p-1}(A) \\ &\longrightarrow \hat{h}_I^p(X, A) \longrightarrow \hat{h}^p(X) \longrightarrow \hat{h}^p(A) \\ &\longrightarrow h^{p+2}(X, A) \longrightarrow h^{p+2}(X) \longrightarrow h^{p+2}(A) \longrightarrow \cdots \end{aligned}$$

3.2. Relative differential cohomology of type II. This definition contains the previous one as a particular case, and generalizes to any cohomology theory the second one of the previous section. It is the most interesting for applications. In the groups of type I we require that the differential function is vanishing on A . For Deligne cohomology, in the groups of type II the differential class on Y must be topologically trivial, and a specific trivialization is chosen as a part of the data, therefore the class on Y is represented by $(1, 0, \dots, 0, \rho)$. Here we generalize this requirement, replacing the representatives of the form $(1, 0, \dots, 0, \rho)$ with the ones of the form $(c_{e_n}, \chi(\rho), d\rho)$. Moreover, for type I a homotopy must be constant on A , i.e. it must be of the form $(c_{e_n}, 0, 0)$ on $A \times I$: here we require again that it is constant, i.e. that it is of the form $(c_{e_n}, \pi^*\chi(\rho), \pi^*d\rho)$ on $A \times I$, where $\pi : A \times I \rightarrow A$ is the projection. The following definition is therefore the natural generalization of 3.2.

Definition 3.3. *If (X, A) is a pair of smooth manifolds (even with boundary, in which case A is a neat submanifold), (Y, y_0) a topological space with marked point, V^{\bullet} a graded real vector space and $\kappa_n \in C^n(Y, y_0, V^{\bullet})$ a real singular cocycle, a relative differential function of type II from (X, A) to (Y, y_0, κ_n) is a quadruple (f, h, ω, ρ) such that:*

- (f, h, ω) is a differential function $(f, h, \omega) : X \rightarrow (Y, \kappa_n)$;
- $\rho \in \Omega^{n-1}(A; V^{\bullet})$;
- $(f, h, \omega)|_A = (c_{y_0}, \chi(\rho), d\rho)$.

Moreover, a homotopy between two relative differential functions of type II (f_0, h_0, ω, ρ) and (f_1, h_1, ω, ρ) is a relative differential function $(F, H, \pi^*\omega, \pi^*\rho) : (X \times I, A \times I) \rightarrow (Y, y_0, \kappa_n)$, such that F is a homotopy between f_0 and f_1 , $H|_{(X \times \{i\}, A \times \{i\})} = h_i$ for $i = 0, 1$, and $\pi : X \times I \rightarrow X$ is the natural projection.

The group $\hat{h}_{II}^n(X, A)$ is defined in the following way:

- as a set, $\hat{h}_{II}^n(X, A)$ contains the homotopy classes of relative differential functions of type II $(f, h, \omega, \rho) : (X, A) \rightarrow (E_n, e_n, \iota_n)$.
- The sum is defined by:

$$(15) \quad [(f, h, \omega, \rho)] + [(g, k, \xi, \psi)] := [(\alpha_n \circ (f, g), h + k + (f, g)^* A_{n-1}, \omega + \xi, \rho + \psi)].$$

- The first Chern class and the curvature are defined as $I[(f, h, \omega)] := [f]$ and $R[(f, h, \omega)] := \omega$, and the map $a : \Omega^{\bullet-1}(X, A; \mathfrak{h}_{\mathbb{R}}^{\bullet}) / \text{Im}(d) \rightarrow \hat{h}_{II}^{\bullet}(X, A)$ (v. [5]) is defined as $a[\rho] := [c_{e_n}, \chi(\rho), d\rho, \rho|_A]$, for c_{e_n} the constant map whose value is the marked point e_n .
- The S^1 -integration map and the product are defined similarly to the absolute case.

The corresponding long exact sequence is the following:

$$(16) \quad \begin{aligned} \dots &\longrightarrow \hat{h}_{II}^{p-1}(X, A) \longrightarrow \hat{h}_{II}^{p-1}(X) \longrightarrow \hat{h}^{p-1}(A) \\ &\longrightarrow \hat{h}_{II}^p(X, A) \longrightarrow \hat{h}^p(X) \longrightarrow h^{p+1}(A) \\ &\longrightarrow h^{p+2}(X, A) \longrightarrow h^{p+2}(X) \longrightarrow h^{p+2}(A) \longrightarrow \dots \end{aligned}$$

There is a natural embedding generalizing (7):

$$(17) \quad \varphi_{I,II} : \hat{h}_I^p(X, A) \hookrightarrow \hat{h}_{II}^p(X, A)$$

which actually extends to a morphism of exact sequences from (14) to (16).

3.3. Relative differential cohomology of type III. This definition generalizes to any cohomology theory the third one of the previous section. It is less interesting than the previous ones, but we put it anyway for completeness. We must require that a relative class is topologically trivial on Y , with no reference to any specific trivialization of the differential class.

Definition 3.4. *If (X, A) is a pair of smooth manifolds (even with boundary, in which case A is a neat submanifold), (Y, y_0) a topological space with marked point, V^{\bullet} a graded real vector space and $\kappa_n \in C^n(Y, y_0, V^{\bullet})$ a real singular cocycle, a relative differential function of type III from (X, A) to (Y, y_0, κ_n) is a triple (f, h, ω) such that:*

- (f, h, ω) is a differential function $(f, h, \omega) : X \rightarrow (Y, \kappa_n)$;
- $f|_A = c_{y_0}$.

Homotopies are defined as in the previous cases.

Even the group structure on $\hat{h}_{III}^n(X, A)$ is defined as in the previous cases. The corresponding long exact sequence is the following:

$$(18) \quad \begin{aligned} \dots &\longrightarrow \hat{h}_{II}^{p-1}(X, A) \longrightarrow \hat{h}_{II}^{p-1}(X) \longrightarrow h^p(A) \\ &\longrightarrow \hat{h}_{III}^p(X, A) \longrightarrow \hat{h}^p(X) \longrightarrow h^{p+1}(A) \\ &\longrightarrow h^{p+2}(X, A) \longrightarrow h^{p+2}(X) \longrightarrow h^{p+2}(A) \longrightarrow \dots \end{aligned}$$

There is a natural surjective map generalizing (9):

$$(19) \quad \varphi_{II,III} : \hat{h}_{II}^p(X, A) \rightarrow \hat{h}_{III}^p(X, A)$$

which actually extends to a morphism of exact sequences from (16) to (18).

3.4. Relative differential cohomology of type IV. In the group $C^\bullet(X, \mathbb{R})$ we consider the subgroup $Z_{\text{ch}}^\bullet(X, \mathbb{R})$ containing the cocycles that represent a class belonging to the image of the Chern character of the theory h^\bullet we are considering (for ordinary cohomology such a subgroup is generated by $Z^\bullet(X, \mathbb{Z})$ and the real coboundaries). We can consider a definition of differential function analogous to the usual one, with the only difference that, in the triple (f, h, ω) , we think of h as an element of $C^{n-1}(X, \mathbb{R})/Z_{\text{ch}}^{n-1}(X, \mathbb{R})$ instead of $C^{n-1}(X, \mathbb{R})$. This has no effects on differential cohomology, since a class $[(c_{e_n}, h, 0)]$ is vanishing if and only if $h \in Z_{\text{ch}}^\bullet(X, \mathbb{R})$. Nevertheless, this quotient has some effects on the relative groups: we consider a definition analogous to 3.2, where now $h|_A = 0$ means that $h|_A \in Z_{\text{ch}}^\bullet(A, \mathbb{R})$.

Definition 3.5. *If (X, A) is a pair of smooth manifolds (even with boundary, in which case A is a neat submanifold), (Y, y_0) a topological space with marked point, V^\bullet a graded real vector space and $\kappa_n \in C^n(Y, y_0, V^\bullet)$ a real singular cocycle, a relative differential function of type IV from (X, A) to (Y, y_0, κ_n) is a triple $(f, [h], \omega)$ such that:*

- $(f, [h], \omega)$ is a differential function $(f, [h], \omega) : X \rightarrow (Y, \kappa_n)$;
- $(f, [h], \omega)|_A = (c_{y_0}, [0], 0)$.

Homotopies are defined as in the previous cases.

With this definition we get a long exact sequence all made by differential cohomology groups, while the ones we obtained up to now contain some topological groups and some differential extensions. The sequence is the following:

$$\begin{aligned}
 (20) \quad & \cdots \longrightarrow \hat{h}_{VI}^{p-1}(X, A) \longrightarrow \hat{h}^{p-1}(X) \longrightarrow \hat{h}^{p-1}(A) \\
 & \longrightarrow \hat{h}_{IV}^p(X, A) \longrightarrow \hat{h}^p(X) \longrightarrow \hat{h}^p(A) \\
 & \longrightarrow \hat{h}_{IV}^{p+1}(X, A) \longrightarrow \hat{h}^{p+1}(X) \longrightarrow \hat{h}^{p+1}(A) \longrightarrow \cdots
 \end{aligned}$$

4. THE EXACT SEQUENCES

We now construct the Bockstein maps of the long exact sequences stated in the previous section, and we prove the exactness in each case.

4.1. Homotopy extension property. In the following we will need the homotopy extension property for differential functions. We call $\text{Cyl } A := A \times I$ and, for a pair (X, A) , we call $\text{Cyl}(X, A)$ the union $X \cup \text{Cyl } A$ identifying $A \subset X$ with $A \times \{0\} \subset \text{Cyl } A$. In general $\text{Cyl}(X, A)$ is not a manifold, nevertheless we will deal with differential functions $(f, h, \omega) : \text{Cyl}(X, A) \rightarrow (E_n, \iota_n)$, requiring that f and h are defined on $\text{Cyl}(X, A)$ and that $\omega|_{\text{Cyl } A} = \pi_A^*(\omega|_A)$, for $\pi_A : A \times I \rightarrow A$ the projection. In other words, (f, h, ω) is the union of a differential function on X and a homotopy on $A \times I$. The same consideration applies if we iterate the cylinder, for example considering $\text{Cyl}(\text{Cyl}(X, A), \text{Cyl } A)$, and so on. In general, every time that a cylinder appears, we suppose the curvature to be equal to the pull-back of the one of the base via the projection. We call “manifold with cylinders” a space obtained in this way.

Lemma 4.1. *Given a pair of smooth manifolds with cylinders (X, A) , the homotopy extension property holds for differential functions, i.e. given $(f, h, \omega) : \text{Cyl}(X, A) \rightarrow (Y, \kappa_n)$ there exists a homotopy $(F, H, \pi_X^*(\omega|_X)) : X \times I \rightarrow (Y, \kappa_n)$ extending (f, h, ω) .*

Proof: Since a pair of smooth manifolds with cylinders is a CW-pair, the homotopy extension property (in the usual sense) holds, therefore we can extend f to $F : X \times I \rightarrow Y$. Since F is homotopic to $f \circ \pi_X$, we have $[\chi^n(\pi_X^*(\omega|_X)) - F^*\iota_n] = \pi_X^*[\chi^n(\omega|_X) - f^*\iota_n] = [\pi_X^*(\delta(h|_X))] = 0$, hence there exists $H' \in C^{n-1}(X \times I; \mathfrak{h}_{\mathbb{R}}^\bullet)$ such that $\delta H' = \chi^n(\pi^*(\omega|_X)) - F^*\iota_n$. The homotopy extension property is equivalent to $\text{Cyl}(X, A)$ being a retract of $X \times I$. Let r be such a retraction and $K := r^*(h - H'|_{X \cup \text{Cyl} A})$: one has $\delta K = 0$. We define $H := H' + K$. Then $\delta H = \chi^n(\pi^*(\omega|_X)) - F^*\iota_n$ and $H|_{\text{Cyl}(X, A)} = h$. \square

4.2. Type I. The exactness of (14) has been already proven in [14]. In particular, the relative flat theory $\hat{h}_{\text{fl}}^p(X, A)$ is by definition the subgroup of $\hat{h}_I^p(X, A)$ made by classes with vanishing curvature, and, being a cohomology theory, it defines a long exact sequence. Hence the Bockstein map of (14) is defined as the composition of the Bockstein map of the flat theory $\hat{h}_{\text{fl}}^{p-1}(A) \rightarrow \hat{h}_{\text{fl}}^p(X, A)$ with the immersion $\hat{h}_{\text{fl}}^p(X, A) \hookrightarrow \hat{h}_I^p(X, A)$.

We define the relative groups of type I in an alternative way, which will be suitable to be generalized to type II in order to construct the Bockstein map of (16). When we consider the relative Deligne cohomology, a relative class of this type is represented by a cocycle on X together with a geometrical trivialization of its restriction to A [9]. Here we can repeat an analogous construction, considering the trivialization as a homotopy of differential functions.

Definition 4.1. *Given a differential function $(f, h, \omega) : X \rightarrow (E_n, \iota_n)$:*

- *a geometric trivialization of (f, h, ω) is a homotopy $(F, H, \pi^*\omega) : X \times I \rightarrow (E_n, \iota_n)$ between (f, h, ω) and the trivial function $(c_{e_n}, 0, 0)$;*
- *a homotopy of geometric trivializations between $(F, H, \pi^*\omega)$ and $(G, K, \pi^*\omega)$ is a homotopy of differential functions $(\Phi, \mathcal{H}, \pi_2^*\omega) : X \times I \times I \rightarrow (E_n, \iota_n)$ relative to $X \times \{0, 1\}$.*

It follows that $\omega|_{X \times I}$ is constant with respect to I , therefore $\omega|_{X \times I} = 0$, in particular $\omega = 0$. For a homotopy, being relative to $X \times \{0, 1\}$ means being equal to $(f \circ \pi_X, \pi_X^*h, 0)$ on $X \times \{0\} \times I$ and equal to $(c_{e_n}, 0, 0)$ on $X \times \{1\} \times I$.

Definition 4.2. *The group $\hat{h}_I^n(X, A)$ contains the homotopy classes of differential functions $(f, h, \omega) : \text{Cyl}(X, A) \rightarrow (E_n, \iota_n)$ such that $(f, h, \omega)|_{\text{Cyl} A}$ is a geometric trivialization of $(f, h, \omega)|_A$. A homotopy between two such functions is a homotopy of differential functions $(F, H, \pi^*\omega) : (\text{Cyl}(X, A)) \times I \rightarrow (E_n, \iota_n)$ such that $(F, H, \pi^*\omega)|_{A \times \{1\} \times I} = (c_{e_n}, 0, 0)$.*

It follows that $\omega|_{\text{Cyl} A} = 0$, in particular $\omega|_A = 0$. We remark that, even if by definition $(f, h, \omega)|_A = (c_{e_n}, 0, 0)$, a homotopy $(F, H, \pi^*\omega)$ does *not* restrict in general to a homotopy of geometric trivializations on $A \times I \times I$, since it must be constant only on $A \times \{1\} \times I$, not on $A \times \{0\} \times I$. The group structure on $\hat{h}_I^n(X, A)$ is defined using a formula analogous to (13) on the space $\text{Cyl}(X, A)$.³ We can show that $\hat{h}_I^n(X, A) \simeq \hat{h}_I^n(X, A)$ canonically. There is a natural morphism $\varphi : \hat{h}_I^n(X, A) \rightarrow \hat{h}_I^n(X, A)$: given $[(f, h, \omega)] \in \hat{h}_I^n(X, A)$, since by definition $(f, h, \omega)|_A = (c_{e_n}, 0, 0)$, we extend (f, h, ω) to $\text{Cyl}(X, A)$ defining $(f, h, \omega)|_{\text{Cyl} A} = (c_{e_n}, 0, 0)$. It is easy to verify that φ is well-defined up to homotopy and that respects the sum.

³We cannot say that the group structure is induced from the one on $\hat{h}^n(\text{Cyl}(X, A))$ because we only consider homotopies which are trivial on $A \times \{1\} \times I$, therefore we do not get a subgroup of $\hat{h}^n(\text{Cyl}(X, A))$ in general.

Theorem 4.2. *The morphism $\varphi : \hat{h}_I^n(X, A) \rightarrow \hat{h}_{I'}^n(X, A)$ is an isomorphism.*

Proof:

Injectivity. Let $\varphi([(f, h, \omega)]) = 0$. Then there exists a homotopy $(F, H, \pi^*\omega)$ between (f, h, ω) extended to $\text{Cyl}(X, A)$ and $(c_{e_n}, 0, 0)$ such that $(F, H, \pi^*\omega)|_{A \times \{1\} \times I} = (c_{e_n}, 0, 0)$. We now apply the homotopy extension property to the pair $(X \times \{0\} \times I, A \times \{0\} \times I)$, extending $(F, H, \pi^*\omega)$ to a function $(F', H', \pi^*\omega) : X \times I \times I \rightarrow (E_n, \iota_n)$ such that $(F', H', \pi^*\omega)|_{A \times I \times \{0\}}$, $(F', H', \pi^*\omega)|_{A \times \{1\} \times I}$ and $(F', H', \pi^*\omega)|_{A \times I \times \{1\}}$ are all equal to $(c_{e_n}, 0, 0)$. Therefore, composing the homotopies $(F', H', \pi^*\omega)|_{X \times I \times \{0\}}$, $(F', H', \pi^*\omega)|_{X \times \{1\} \times I}$ and $(F', H', \pi^*\omega)|_{X \times I \times \{1\}}$ we get a homotopy between (f, h, ω) and $(c_{e_n}, 0, 0)$ which is trivial on A , thus we get a homotopy of relative differential functions between $(f, h, \omega) : (X, A) \rightarrow (E_n, e_n, \iota_n)$ and $(c_{e_n}, 0, 0)$: this proves that $[(f, h, \omega)] = 0$ in $\hat{h}_I^n(X, A)$.

Surjectivity. Let us consider $[(f, h, \omega)] \in \hat{h}_{I'}^n(X, A)$, in particular $(f, h, \omega) : \text{Cyl}(X, A) \rightarrow (E_n, \iota_n)$ and $(f, h, \omega)|_{A \times I} = (c_{e_n}, 0, 0)$. We now construct a homotopy $(F, H, 0) : A \times I \times I \rightarrow (E_n, \iota_n)$ between $(f, h, \omega)|_{A \times I}$ and $(c_{e_n}, 0, 0)$, such that $(F, H, 0)|_{A \times \{1\} \times I} = (c_{e_n}, 0, 0)$. We define:

$$F(a, u, t) := \begin{cases} f(a, u + t) & u + t \leq 1 \\ f(a, 1) & u + t \geq 1. \end{cases}$$

In this way $F(a, u, 0) = f(a, u)$ and $F(a, u, 1) = F(a, 1, t) = f(a, 1) = e_n$. We call $\pi_{A \times I \times \{0\}} : A \times I \times I \rightarrow A \times I \times \{0\}$ the projection, and similarly for the other cases. We have that F is homotopic to $f \circ \pi_{A \times I \times \{0\}}$, therefore $[F^* \iota_n] = -\pi_{A \times I \times \{0\}}^* [\delta^{n-1} h] = 0$, hence there exists $H'' \in C^{n-1}(A \times I \times I, \mathfrak{h}_{\mathbb{R}}^\bullet)$ such that $\delta H'' = -F^* \iota_n$. We must now replace H'' by a cocycle H such that $H|_{A \times I \times \{0\}} = h$ and $H|_{A \times I \times \{1\}} = H|_{A \times \{1\} \times I} = 0$. We do it in three steps:

- We call $K'' := \pi_{A \times \{1\} \times I}^*(H''|_{A \times \{1\} \times I})$. Then $\delta K'' = 0$, hence we define $H' = H'' - K''$. In this way $\delta H' = -F^* \iota_n$ and $H'|_{A \times \{1\} \times I} = 0$.
- We call $K' := \pi_{A \times I \times \{0\}}^*(h - H'|_{A \times I \times \{0\}})$: then $\delta K' = 0$. For $H := H' + K'$ we get $\delta H = -F^* \iota_n$ and $H|_{A \times I \times \{0\}} = h$, keeping $H|_{A \times \{1\} \times I} = 0$.
- Up to now $H|_{A \times I \times \{1\}} = h'$ with $\delta h' = 0$ and $h'|_{A \times \{1\} \times \{1\}} = 0$. Since $A \times I$ retracts by deformation to A and the differential function we started from is trivial on A , the cohomology class of h' vanishes, i.e. $h' = \delta k'$. It follows that $(c_{e_n}, \delta k', 0)$ is homotopic to $(c_{e_n}, 0, 0)$ and, by [14, Lemma 2.3], we can choose the homotopy to be constant (hence vanishing) on $A \times \{1\}$. We compose $(F, H, 0)$ with this homotopy, and for simplicity we still call the result $(F, H, 0)$. Now $H|_{A \times I \times \{1\}} = 0$.

Thanks to the homotopy extension property of the pair $(\text{Cyl}(X, A), \text{Cyl } A)$, we extend $(F, H, 0)$ to $(F, H, \pi^*\omega) : \text{Cyl}(X, A) \times I \rightarrow (E_n, \iota_n)$. By construction the function $(F, H, \pi^*\omega) : \text{Cyl}(X, A) \times \{1\} \rightarrow (E_n, \iota_n)$ represents a class lying in the image of φ and, considering definition 4.2, it is homotopic to (f, h, ω) : it follows that the class $[(f, h, \omega)]$ we started from belongs to the image of φ . \square

Remark: Given a class $[(f, h, \omega)] \in \hat{h}_{I'}^n(X, A)$, by the homotopy extension property applied to (X, A) we can extend (f, h, ω) to $(\Phi, \mathcal{H}, \pi_X^*(\omega|_X)) : X \times I \rightarrow (E_n, \iota_n)$. The restriction to $\{1\} \times X$ is a differential function $(g, h, \omega|_X) : X \rightarrow (E_n, \iota_n)$. Since the restriction to A is $(c_{e_n}, 0, 0)$ by construction, we get a class $[(g, h, \omega|_X)] \in \hat{h}_I^n(X, A)$. We claim that $\varphi[(g, h, \omega|_X)] = [(f, h, \omega)]$. In fact, in the proof of surjectivity, we can extend $(F, H, 0)$ to the whole $\text{Cyl } X \times I$ applying the homotopy extension property

to the pair $(\text{Cyl } X, \text{Cyl } A)$ instead of $(\text{Cyl}(X, A), \text{Cyl } A)$. In this way the homotopies $(F, H, \pi^*\omega)|_{X \times \{1\} \times I}$ and $(F, H, \pi^*\omega)|_{X \times I \times \{1\}}$ compose to a homotopy (with respect to def. 3.2) between $(g, h, \omega|_X)$ and $(F, H, \pi^*\omega)|_{X \times \{0\} \times \{1\}}$, the latter being by construction a representative of $\varphi^{-1}[(f, h, \omega)]$. \square

The Bockstein map of (14) can be visualized in this way: given $\alpha \in \hat{h}_{\text{fl}}^{n-1}(A, x_0)$, we consider the suspension isomorphism in the flat theory and we get $\tilde{\alpha} \in \hat{h}_{\text{fl}}^n(SA, *)$. For $p : A \times I \rightarrow SA$ the natural projection, such that $p(A \times \{0, 1\}) = \{*\}$, we consider $p^*\tilde{\alpha}$ and we extend it trivially on $\text{Cyl}(X, A)$. The class we get is the image of α in $\hat{h}_{\text{fl}}^n(X, A)$. Thanks to the previous theorem, we get a corresponding class in $\hat{h}_I^n(X, A)$.

4.3. Type II. We now adapt the construction of the previous paragraph to the groups of type II, in order to define the Bockstein map of (16) and show the exactness of the latter. When we consider the relative Deligne cohomology, a relative class of this type is represented by a cocycle on X together with a strong topological trivialization of its restriction to A [9]. Here we consider the trivialization as a suitable homotopy of differential functions.

Definition 4.3. *Given a differential function $(f, h, \omega) : X \rightarrow (E_n, \iota_n)$:*

- *a strong topological trivialization of (f, h, ω) is a homotopy $(F, H, \pi^*\omega) : X \times I \rightarrow (E_n, \iota_n)$ between (f, h, ω) and a function of the form $(c_{e_n}, \chi(\rho), d\rho)$;*
- *a homotopy of strong topological trivialization between $(F, H, \pi^*\omega)$ and $(G, K, \pi^*\omega)$ is a homotopy of differential functions $(\Phi, \mathcal{H}, \pi_2^*\omega) : X \times I \times I \rightarrow (E_n, \iota_n)$ relative to $X \times \{0, 1\}$.*

It follows that $\omega|_{X \times I}$ is constant with respect to I , therefore $\omega|_{X \times I} = \pi_X^*(d\rho)$, in particular $\omega = d\rho$. For a homotopy, being relative to $X \times \{0, 1\}$ means being equal to $(f \circ \pi_X, \pi_X^*h, \pi_X^*d\rho)$ on $X \times \{0\} \times I$ and equal to $(c_{e_n}, \chi(\pi_X^*\rho), \pi_X^*d\rho)$ on $X \times \{1\} \times I$.

Definition 4.4. *The group $\hat{h}_{II}^n(X, A)$ contains the homotopy classes of differential functions $(f, h, \omega) : \text{Cyl}(X, A) \rightarrow (E_n, \iota_n)$ such that $(f, h, \omega)|_{\text{Cyl}A}$ is a strong topological trivialization of $(f, h, \omega)|_A$. A homotopy between two such functions, whose trivialization on A is $(c_{e_n}, \chi(\rho), d\rho)$, is a homotopy of differential functions $(F, H, \pi^*\omega) : (\text{Cyl}(X, A)) \times I \rightarrow (E_n, \iota_n)$ such that $(F, H, \pi^*\omega)|_{A \times \{1\} \times I} = (c_{e_n}, \chi(\pi_A^*\rho), \pi_A^*d\rho)$.*

It follows that $\omega|_{\text{Cyl}A} = \pi_A^*(d\rho)$, in particular $\omega|_A = d\rho$. We remark that, if $(f, h, \omega)|_A = (c_{e_n}, \chi(\rho), d\rho)$, a homotopy $(F, H, \pi^*\omega)$ does not restrict in general to a homotopy of strong topological trivializations on $A \times I \times I$, since it must be constant only on $A \times \{1\} \times I$, not on $A \times \{0\} \times I$. The group structure on $\hat{h}_{II}^n(X, A)$ is defined using a formula analogous to (15) on the space $\text{Cyl}(X, A)$. We can show that $\hat{h}_{II}^n(X, A) \simeq \hat{h}_I^n(X, A)$ canonically. There is a natural morphism $\varphi : \hat{h}_{II}^n(X, A) \rightarrow \hat{h}_I^n(X, A)$: given $[(f, h, \omega)] \in \hat{h}_{II}^n(X, A)$, since by definition $(f, h, \omega)|_A = (c_{e_n}, \chi(\rho), d\rho)$, we extend (f, h, ω) to $\text{Cyl}(X, A)$ defining $(f, h, \omega)|_{\text{Cyl}A} = (c_{e_n}, \chi(\pi_A^*\rho), \pi_A^*d\rho)$. It is easy to verify that φ is well-defined up to homotopy and that respects the sum.

Theorem 4.3. *The morphism $\varphi : \hat{h}_{II}^n(X, A) \rightarrow \hat{h}_I^n(X, A)$ is an isomorphism.*

Proof: About the injectivity, the proof of theorem 4.2 applies, since, if $\varphi([(f, h, \omega)]) = 0$, the trivialization on A must be $(c_{e_n}, 0, 0)$ (i.e. $\rho = 0$). About the surjectivity, the proof of theorem 4.2 can be adapted considering the homotopy $(F, H, \pi_{\text{Cyl } A}^* \pi_A^* d\rho) : A \times I \times I \rightarrow (E_n, \iota_n)$ instead of $(F, H, 0)$.⁴ \square

A remark analogous to the one after theorem 4.2 still holds. We can now construct the Bockstein map of (16). We start from $[(f, h, \rho)] \in \hat{h}^{n-1}(A)$, and, by analogy with the Deligne cohomology, we must get a class in $\hat{h}_{II}^n(X, A)$ whose restriction to A is $(c_{e_n}, \chi(-\rho), 0)$. Nevertheless, we cannot simply extend such a restriction to X , because the extension is not unique. Actually, if the Bockstein map were defined in this way, only the curvature ρ of $[(f, h, \rho)]$ would be meaningful, therefore the kernel would be $\hat{h}_{\text{fl}}^{n-1}(A)$, not the image of $\hat{h}_{\text{fl}}^{n-1}(X)$. We thus need a different construction, passing through the group $\hat{h}_{II'}^n(X, A)$.

Briefly the idea is the following. We call $\pi_1 : S^1 \times A \rightarrow A$ the projection and we fix a marked point on S^1 , e.g. $1 \in S^1 \subset \mathbb{C}$. Given $[(f, h, \rho)] \in \hat{h}^{n-1}(A)$, we consider the unique class $[(F, H, dt \wedge \pi_1^* \rho)] \in \hat{h}_I^n(S^1 \times A, \{1\} \times A)$ whose integral over S^1 is $[(f, h, \rho)]$, and, identifying the pair $(S^1 \times A, \{1\} \times A)$ with the pair $(I \times A, \{0, 1\} \times A)$, we define $\beta^{n-1}[(f, h, \rho)] := [(F, H - \chi(t \cdot \pi_A^* \rho), 0)] \in \hat{h}_{II'}^n(X, A)$, the extension to X being the trivial one. We describe this construction in more detail.

- Given $[(f, h, \rho)] \in \hat{h}^{n-1}(A)$, thanks to [14, Lemma 4.4] there exists a class $[(F, H, dt \wedge \pi_1^* \rho)] \in \hat{h}_I^n(S^1 \times A, \{1\} \times A)$ such that $\int_{S^1} [(F, H, dt \wedge \pi_1^* \rho)] = [(f, h, \rho)]$. Such a class is unique: if we choose another one, the difference is a flat class $[(F', H', 0)] \in \hat{h}_{\text{fl}}^n(S^1 \times A, \{1\} \times A) \simeq \hat{h}_{\text{fl}}^{n-1}(A)$, the isomorphism being given by the S^1 -integration.⁵ Since $\int_{S^1} [(F', H', 0)] = 0$, we get $[(F', H', 0)] = 0$.
- Composing with the pull-back via the projection $p : (I \times A, \{0, 1\} \times A) \rightarrow (S^1 \times A, \{1\} \times A)$, we get a class represented by a differential function on $I \times A$ that we still call $(F, H, dt \wedge \pi_A^* \rho)$, whose restriction to $\{0, 1\} \times A$ is 0.
- We define the following differential function on $I \times A$:

$$(21) \quad (F, H - \chi(t \cdot \pi_A^* \rho), 0).$$

Such a function is well-defined, since, by construction, $\delta H = \chi(dt \wedge \pi_A^* \rho) - F^* \iota_n = \chi(d(t \cdot \pi_A^* \rho)) - F^* \iota_n = \delta \chi(t \cdot \pi_A^* \rho) - F^* \iota_n$, hence $\delta(H - \chi(t \cdot \pi_A^* \rho)) = -F^* \iota_n$.

- By construction $(F, H - \chi(t \cdot \pi_A^* \rho), 0)|_{\{1\} \times A} = (c_{e_n}, \chi(-\rho), 0)$ (we recall that $d\rho = 0$ since ρ is a curvature), and $(F, H - \chi(t \cdot \pi_A^* \rho), 0)|_{\{0\} \times A} = (c_{e_n}, 0, 0)$. Hence we extend such a class to $\text{Cyl}(X, A)$ requiring that it is trivial on X , and we get a representative of a class in $\hat{h}_{II'}^n(X, A)$.

We can now verify that the map is well-defined and it is a group homomorphism. We have already pointed out that the class $[(F, H, dt \wedge \pi_1^* \rho)] \in \hat{h}_I^n(S^1 \times A, \{1\} \times A)$ is unique. If we choose another representative $(F', H', dt \wedge \pi_1^* \rho)$, by definition there exists a homotopy $(\Phi, \mathcal{H}, dt \wedge \pi^* \rho)$ defined on $I \times S^1 \times A$ which is constant on $\{1\} \times A$, therefore, pulling back to $I \times A$, we get a homotopy defined on $I \times I \times A$ which is constant on $\{0, 1\} \times A$. Considering $(\Phi, \mathcal{H} - \chi(t \cdot \pi_A^* \rho), 0)$ we get a homotopy defined on $I \times I \times A$ which is constant

⁴Moreover, the definition of K'' becomes $K'' := \pi_{A \times \{1\} \times I}^* (H'' - \chi(\rho))$.

⁵In particular, calling $A_+ := A \sqcup \{\infty\}$ and S the suspension, we have that $\hat{h}_{\text{fl}}^n(S^1 \times A, \{1\} \times A) \simeq \tilde{\hat{h}}_{\text{fl}}^n((S^1 \times A)/(\{1\} \times A)) \simeq \tilde{\hat{h}}_{\text{fl}}^n(S(A_+)) \simeq \tilde{\hat{h}}_{\text{fl}}^{n-1}(A_+) \simeq \hat{h}_{\text{fl}}^{n-1}(A)$.

and equal to $(c_{e_n}, 0, 0)$ on $\{0\} \times A$, and constant and equal to $(c_{e_n}, \chi(-\rho), 0)$ on $\{1\} \times A$. Therefore the two classes $(F, H - \chi(t \cdot \pi_A^* \rho), 0)$ and $(F', H' - \chi(t \cdot \pi_A^* \rho), 0)$ define the same class in $\hat{h}_{II'}^n(X, A)$. In order to show that it is a group homomorphism, we notice that the class represented by (21) is equal to $[(F, H, dt \wedge \pi_A^* \rho)] - [(c_{e_n}, \chi(t \cdot \pi_A^* \rho), dt \wedge \pi_1^* \rho)]$, therefore the Bockstein map we constructed is equal to the difference of two homomorphisms, hence it is a homomorphism.⁶

. We can now prove the exactness of (16):

- *Exactness in $\hat{h}^{n-1}(A)$.* Given $[(f, h, \rho)] \in \hat{h}^{n-1}(A)$, if $\beta^{n-1}[(f, h, \rho)] = 0$ then $\rho = 0$, since the restriction of (21) to $\{1\} \times A$ is $(c_{e_n}, -\chi(\rho), 0)$. Therefore the kernel of β^{n-1} is contained in the flat part $\hat{h}_{\text{fl}}^{n-1}(A)$, hence the exactness follows from the one of (14).
- *Exactness in $\hat{h}_{II'}^n(X, A)$.* If a class belongs to the image of the Bockstein map, it follows from the previous construction that its restriction to X is trivial. Viceversa, let us consider $[(F, H', \omega)] \in \hat{h}_{II'}^n(X, A)$ such that $[(F, H', \omega)]|_X$ is trivial. By the homotopy extension property applied to the pair $(\text{Cyl}(X, A), X \cup (\{1\} \times A))$, we can suppose that $(F, H', \omega)|_X = (c_{e_n}, 0, 0)$. By definition of $\hat{h}_{II'}^n(X, A)$, we have that $(F, H', \omega)|_{\{1\} \times A} = (c_{e_n}, -\chi(\rho), 0)$, therefore we define $H := H' + \chi(t \cdot \pi_A^* \rho)$, so that (F, H', ω) takes the form (21). It follows that the differential function $(F, H, dt \wedge \pi_A^* \rho)$ is well-defined on $I \times A$, since $\delta H = \delta H' + \chi(dt \wedge \pi_A^* \rho) = -F^* \iota_n + \chi(dt \wedge \pi_A^* \rho)$. Being $(F, H, dt \wedge \pi_A^* \rho)|_{\partial I \times A} = (c_{e_n}, 0, 0)$, it represents a class in $\hat{h}_I^n(I \times A, \partial I \times A) \simeq \hat{h}_I^n(S^1 \times A, \{1\} \times A)$, hence we obtain $[(F, H, dt \wedge \pi_1^* \rho)] \in \hat{h}_I^n(S^1 \times A, \{1\} \times A)$. Integrating it over S^1 we get a class $[(f, h, \rho)] \in \hat{h}^{n-1}(A)$ such that $\beta^{n-1}[(f, h, \rho)] = [(F, H', \omega)]$.⁷
- *Exactness in $\hat{h}^n(X)$.* By construction the restriction to A of a class in $\hat{h}_{II'}^n(X, A)$ is topologically trivial. Viceversa, given a class $[(f, h, \omega)] \in \hat{h}^n(X)$ which is topologically trivial on A , by the homotopy extension property we can suppose that $(f, h, \omega)|_A = (c_{e_n}, \chi(\rho), d\rho)$. It follows that such a class is the image of a class $[(f, h, \omega)] \in \hat{h}_{II'}^n(X, A)$.

4.4. Type III. We briefly consider the groups of type III in order to complete the picture. When we consider the relative Deligne cohomology, a relative class of this type is represented by a cocycle on X together with a topological trivialization of its restriction to A . Here we consider the trivialization as a suitable homotopy of differential functions.

Definition 4.5. *Given a differential function $(f, h, \omega) : X \rightarrow (E_n, \iota_n)$ a topological trivialization of (f, h, ω) is a homotopy $F : X \times I \rightarrow E_n$ between f and c_{e_n} .*

Definition 4.6. *The group $\hat{h}_{III'}^n(X, A)$ contains the homotopy classes of differential functions $(f, h, \omega) : X \rightarrow (E_n, \iota_n)$ with a topological trivialization of $f|_A$ extending f to*

⁶In the difference $[(F, H, dt \wedge \pi_1^* \rho)] - [(c_{e_n}, \chi(t \cdot \pi_1^* \rho), dt \wedge \pi_1^* \rho)]$ we can skip the term $(F, c_{e_n})^* A_{n-1}$, since A_{n-1} can be chosen to be vanishing on $E_n \vee E_n = \{e_n\} \times E_n \cup E_n \times \{e_n\}$. This follows from the definition [14], supposing $\alpha_n(x, e_n) = \alpha_n(e_n, x) = x$.

⁷Of course the class $[(f, h, \rho)]$ depends in general on the representative (F, H', ω) chosen, since, when passing to $S^1 \times A$ from $I \times A$, there is not a well-defined push-forward in cohomology. This is not problem, since we must find at least one class in $\hat{h}^{n-1}(A)$ whose image under the Bockstein map is $[(F, H', \omega)]$. Actually, it cannot be unique in general because of exactness of (16).

$\text{Cyl}(X, A)$. A homotopy between two such functions is a homotopy of differential functions $(F, H, \pi^*\omega) : X \times I \rightarrow (E_n, \iota_n)$ extending to a homotopy of homotopies F on $\text{CylCyl}A$ such that $F|_{A \times \{1\} \times I} = c_{e_n}$.

As in the previous cases, there is a natural morphism $\varphi : \hat{h}_{III}^n(X, A) \rightarrow \hat{h}_{III'}^n(X, A)$ which is actually an isomorphism. In order to construct the Bockstein map of (18), we consider a class $[f] \in h^n(A)$ and the corresponding class $[F] \in h^{n+1}(SA)$. We pull-back such a class to $[F] \in h^{n+1}(I \times A, \partial I \times A)$, and we extend F to a trivial differential function on X . In this way we get a class in $\hat{h}_{III'}^n(X, A)$ whose restriction to X is trivial. One can verify that (18) is exact.

4.5. Type IV. Let us consider the group of type II $\hat{h}_{II}^n(X, A)$: we define the subgroup $\hat{h}_{II, \text{ch}}^n(X, A)$ made by those classes $[(f, h, \rho)]$ such that $[(f, h, \rho)]|_A = 0$: by definition this means that $(f, h, \omega)|_A = (c_{e_n}, \chi(\rho), 0)$, with ρ a closed form representing a class belonging to the image of the Chern character. This subgroup is bigger than $\hat{h}_I^n(X, A)$, because in the latter the form ρ must be 0, i.e. $(f, h, \rho)|_A = 0$ as a single differential function, not only as a cohomology class. There is a natural morphism:

$$(22) \quad \varphi : \hat{h}_{II, \text{ch}}^n(X, A) \rightarrow \hat{h}_{IV}^n(X, A).$$

In fact, given a class $[(f, h, \omega)] \in \hat{h}_{II, \text{ch}}^n(X, A)$, which is equal to $(c_{e_n}, \chi(\rho), 0)$ on A , since ρ represents a class belonging to the image of the Chern character we get a class $[(f, [h], \omega)] \in \hat{h}_{IV}^n(X, A)$, which is well-defined: in fact, a homotopy of type II representatives restricts to $(c_{e_n}, \chi(\pi_A^*\rho), 0)$ on $I \times A$, hence, being $(c_{e_n}, [\chi(\pi_A^*\rho)], 0) = (c_{e_n}, [0], 0)$, it also defines a homotopy of type IV representatives.

Moreover, we denote by $\Omega_{\text{ch}}^n(X, \mathfrak{h}_{\mathbb{R}}^\bullet)$ the group of $\mathfrak{h}_{\mathbb{R}}^\bullet$ -valued closed forms of degree n that represent a class belonging to the image of the Chern character. There is a natural map:

$$(23) \quad \psi : \Omega_{\text{ch}}^{n-1}(X, \mathfrak{h}_{\mathbb{R}}^\bullet) \rightarrow \hat{h}_{II, \text{ch}}^n(X, A)$$

defined by $\psi(\rho) = [(c_{e_n}, \chi(\rho), 0)]$.

Lemma 4.4. *The morphism (22) is surjective and its kernel is the image of (23).*

Proof: For the surjectivity, given a class $[(f, [h], \omega)] \in \hat{h}_{IV}^n(X, A)$, we choose a representative (f, h, ω) . Then $h|_A \in Z_{\text{ch}}^{n-1}(A, \mathbb{R})$ and it is cohomologous to a closed differential form, i.e. $h|_A = \chi(\rho) + \delta k$. The function $(c_{e_n}, h|_A, 0)$ is homotopic to $(c_{e_n}, \chi(\rho), 0)$ via a homotopy $(c_{e_n}, \chi(\rho) + \delta K, 0)$ (v. [14, Lemma 2.3]), hence, by the homotopy extension property, we can represent the class we started from as $[(f, [h], \omega)]$ with $(f, h, \omega)|_A = (c_{e_n}, \chi(\rho), 0)$, therefore any class $[(f, [h], \omega)]$ belongs to the image of φ .

We now show that the kernel of (22) is the image of (23). The inclusion $\text{Im} \psi \subset \text{Ker} \varphi$ is an immediate consequence of the definition of $\hat{h}_{IV}^n(X, A)$. Viceversa, let us suppose that $\varphi[(f, h, 0)] = 0$. Then there is a homotopy $(F, H, 0)$ between $(f, h, 0)$ and $(c_{e_n}, k, 0)$ such that $k \in Z_{\text{ch}}^n(X, \mathbb{R})$. Moreover, $h|_A = (c_{e_n}, \chi(\rho), 0)$ and $H|_{A \times I} = (c_{e_n}, \pi^*\chi(\rho) + \delta K, 0)$, the latter because $H|_{A \times I}$ must be a cocycle in $Z_{\text{ch}}^{n-1}(X, \mathbb{R})$, therefore, being A a deformation retract of $A \times I$, it must be cohomologous to $\pi^*\chi(\rho)$. Extending K to the whole X we get a homotopy $(F, H - \delta K, 0)$ between $(f, h, 0)$ and $(c_{e_n}, k', 0)$ such that $k'|_A = \chi(\rho)$. Then k is cohomologous, relatively to A , to a differential form extending ρ to X , hence the class

we started from lies in the image of ψ . \square

In order to prove the exactness of (20), we consider the groups $\overline{h}^p(X, A)$ defined by (10) generalized to any cohomology theory, i.e.:

$$(24) \quad \overline{h}^p(X, A) := \frac{\text{Ker}(\hat{h}_{II}^p(X, A) \rightarrow \hat{h}^p(A))}{\text{Im}(\hat{h}^{p-1}(X) \rightarrow \hat{h}_{II}^p(X, A))}.$$

Theorem 4.5. *There is a natural isomorphism:*

$$(25) \quad \Xi^p : \overline{h}^p(X, A) \xrightarrow{\simeq} \hat{h}_{IV}^p(X, A).$$

Proof: The numerator of (24) corresponds to $\hat{h}_{II, \text{ch}}^n(X, A)$, because the latter is exactly the group of classes of type II vanishing on A . Let us show that the denominator coincides with the image of (23), so that the result follows from lemma 4.4. Because of (16) the kernel of the map $\hat{h}^{p-1}(X) \rightarrow \hat{h}_{II}^p(X, A)$ is made by the image of flat classes on X , therefore, when composing with $\hat{h}^{n-1}(X) \rightarrow \hat{h}^{n-1}(A)$, only the curvature of the original class in X is meaningful. In particular, given $[(f, h, \rho)] \in \hat{h}^{n-1}(A)$, in order to compute the Bockstein map of (16) we consider the unique class $[(F, H, dt \wedge \pi_1^* \rho)] \in \hat{h}_I^n(S^1 \times A, \{1\} \times A)$ whose integral over S^1 is $[(f, h, \rho)]$, and we define $\beta^{n-1}[(f, h, \rho)] := [(F, H - \chi(t \cdot \pi_A^* \rho), 0)] \in \hat{h}_{II'}^n(X, A)$. If $[(f, h, \rho)]$ is the restriction of a class on the whole X , we apply the same procedure to the whole class obtaining a homotopy $(F, H - \chi(t \cdot \pi_X^* \rho), 0)$ on $I \times X$, which restricts on $\text{Cyl}(X, A)$ to a representative of $\beta^{n-1}[(f, h, \rho)]$, and restricts on $\{1\} \times X$ to $(c_{e_n}, -\chi(\rho), 0)$. Considering the proof of theorem 4.3, we can construct a homotopy on $I \times \text{Cyl}(X, A)$ from $\beta^{n-1}(f, h, \rho)$ to a class which is equal to $(c_{e_n}, -\chi(\rho), 0)$ on the whole $\text{Cyl } A$. By construction such a homotopy is equal to $(c_{e_n}, -\chi(\rho), 0)$ on $I \times (\{1\} \times A)$, hence it can be extended also to $I \times (\{1\} \times X)$, being still equal to $(c_{e_n}, \chi(-\rho), 0)$. By the homotopy extension property applied to the pair $(\text{Cyl } X, \text{Cyl}(X, A) \cup (\{1\} \times X))$, we get a homotopy on $I \times I \times X$ that, on $\{1\} \times I \times X$, restricts to a homotopy between a representative of $\beta^{n-1}[(f, h, \rho)]$ in $\hat{h}_{II}^n(X, A)$ (v. remark after theorem 4.2 adapted to type II) and $(c_{e_n}, \chi(-\rho), 0)$. Hence $\beta^{n-1}[(f, h, \rho)] = \psi(-\rho)$. \square

The fact that (20) is exact therefore follows from [9, Theorem 2.1], which we repeat here for completeness with the notation of the present paper.

Theorem 4.6. *The sequence (20) is exact.*

Proof: The Bockstein map β of (20) is induced from the one of (16), that we call β' : the image of β' is contained in the numerator of (24) because of the exactness of (16). By definition of the denominator of (24) the kernel of β is the image of the restriction map $\hat{h}^{p-1}(X) \rightarrow \hat{h}^{p-1}(A)$, thus (20) is exact in $\hat{h}^{p-1}(A)$. The image of β' contains the denominator of (24), therefore the exactness in $\hat{h}_{IV}^p(X, A)$ follows from the one of (16). Finally, in order to prove the exactness in $\hat{h}^p(X)$, we consider the following commutative diagram:

$$\begin{array}{ccc} \hat{h}_I^p(X, A) & \xrightarrow{\eta} & \hat{h}^p(X) \\ \psi \downarrow & \nearrow \nu & \\ \hat{h}_{IV}^p(X, A) & & \end{array}$$

where η is the map appearing in (14), ν the one appearing in (20) and ψ is the composition of the embedding $\hat{h}_I^p(X, A) \rightarrow \hat{h}_{II}^p(X, A)$, whose image is contained in the numerator of (24) because of the exactness of (14), with the projection to the quotient in (24). We show that $\text{Im } \eta = \text{Im } \nu$, so that the exactness of (20) in $\hat{h}^p(X)$ follows from the one of (14). Obviously $\text{Im } \eta \subset \text{Im } \nu$. For the converse, the image of the embedding $\hat{h}_I^p(X, A) \rightarrow \hat{h}_{II}^p(X, A)$ is the subset of classes which are trivial when pulled-back to $\hat{h}^p(A)$, therefore, applying ν to the numerator of (24), we get classes belonging to the kernel of $\hat{h}^p(X) \rightarrow \hat{h}^p(A)$, i.e. to the image of η . \square

5. GENERIC MAP

Up to now we have considered the relative groups of a pair (X, A) , the latter being equivalent to an embedding $i : A \hookrightarrow X$. We can easily generalize the definition to any map of manifolds $\varphi : A \rightarrow X$, repeating all of the previous constructions with the following two remarks:

- the cylinder $\text{Cyl}(X, A)$ is replaced by $\text{Cyl}(\varphi)$, the latter being the union of the two spaces X and $I \times A$ identifying $a \in A \subset X$ with $(\varphi(a), 0)$ (for a pair (X, A) we have $\varphi(a) = a$);
- a restriction of any class on X to A is replaced by the pull-back via φ .

5.1. Relative differential cohomology of type I.

Definition 5.1. A relative differential function of type I from $\varphi : A \rightarrow X$ to (Y, y_0, κ_n) is a triple (f, h, ω) such that:

- (f, h, ω) is a differential function $(f, h, \omega) : X \rightarrow (Y, \kappa_n)$;
- $\varphi^*(f, h, \omega) = (c_{y_0}, 0, 0)$.

Homotopies are defined as in definition 3.1.

The group $\hat{h}_I^n(X, A)$ is defined considering the homotopy classes of relative differential functions. The exact sequence we get is again (14), replacing the restriction to A with the pull-back via φ .

5.2. Relative differential cohomology of type II.

Definition 5.2. If $\varphi : A \rightarrow X$ is a smooth map, (Y, y_0) a topological space with marked point, V^\bullet a graded real vector space and $\kappa_n \in C^n(Y, y_0, V^\bullet)$ a real singular cocycle, a relative differential function of type II from φ to (Y, y_0, κ_n) is a quadruple (f, h, ω, ρ) such that:

- (f, h, ω) is a differential function $(f, h, \omega) : X \rightarrow (Y, \kappa_n)$;
- $\rho \in \Omega^{n-1}(A; V^\bullet)$;
- $\varphi^*(f, h, \omega) = (c_{y_0}, \chi(\rho), d\rho)$.

Moreover, a homotopy between two relative differential functions of type II (f_0, h_0, ω, ρ) and (f_1, h_1, ω, ρ) is a relative differential function $(F, H, \pi^*\omega, \pi^*\rho)$ from $(\varphi \times I)$ to (Y, y_0, κ_n) , such that F is a homotopy between f_0 and f_1 , $H|_{(X \times \{i\})} = h_i$ for $i = 0, 1$, and $\pi : X \times I \rightarrow X$ is the natural projection.

The group $\hat{h}_{II}^n(X, A)$ is defined considering the homotopy classes of relative differential functions. The exact sequence we get is again (16), and the same construction of the Bockstein map applies, remembering the two remarks stated above.

5.3. Relative differential cohomology of type III and IV. They can be generalized like the previous cases. We leave the details to the reader.

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